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Some formal guideposts for the analysis of R&D models of economic growth (preliminary notes)

In what follows we build a framework which embeds different views of the relation between output growth and the generation of new inputs, as may be encountered in R&D growth models. This is done under a number of simplifying assumptions about technology that still enable us to discuss usually neglected issues, such as the role of complementarities and the relation between technological compatibility and knowledge spillovers. The main simplifying assumption is that the service characteristics of final output Y are unchanged throughout, that Y can be either consumed or accumulated in the form of capital and that it is produced by means of intermediate goods and labour. The set of available intermediate goods Q_t changes through time as a result of innovation activities.

Assume the number of service-characteristics types that exist in nature is finite. An intermediate good is a couple $(v, q) \in Z_+^2$. v is the intermediate-good variety, which identifies a class of functions performed by (v, q), that is, a composition of the associated flow of service characteristics. For instance, a particular oil may serve mainly as a propeller, but partly also as a lubricant. The number of varieties at t is N_t. q is the technological level, or generation, to which (v, q) belongs. In principle, we should expect that q has only an ordinal meaning, possibly with the further ordinal implication that later generations of a variety are also more productive. This is not, however, the interpretation we find in the new-growth literature, where the *q's* cease to be simply indexes of novelty, to become indexes leading to a cardinal productivity measure. As we shall see in a moment, the trick is that of assuming that each generation q of any variety v has a well defined productivity effect; for instance that the marginal product of (v, q) is a known time-invariant function of A^q (and possibly other variables), where $A > 1$. This leads to a time invariant production possibility frontier, describing the productive potential of every possible present and future combination of intermediate goods.

There are two types of production activities in the economy. $\{D\}_t$ is the set of activities in existence at t for producing y by means of intermediate inputs and labour; these activities are performed by perfectly competitive firms. This is consistent with the assumption that for the individual firm final-good production is subject to constant returns to scale. $\{B\}_t$ is the set of activities in existence at t for producing intermediate goods by means of the final good, accumulated in the form of capital. Intermediate goods are differentiated and are produced by local monopolists. There are also invention activities, that will be formalised later on. The reason why firms in the intermediate-good sector can not be perfectly competitive is quite robust (Arrow (1987) and (1998), Romer (1990)). The right to produce a new intermediate good involves an innovation cost that represents a fixed cost, because once the knowledge to produce a unit of a new good is acquired, it can be applied to the production of an indefinite number of units. If intermediate-good production is otherwise subject to constant variable costs, we are faced with a clear case of increasing returns.

D(t) is a an activity available at time t for producing the final good. The inputs of activity ***D(t)*** form a list $(L_{D(t)},\boldsymbol{X}_{D(t)})$, where $L_{A(t)}$ is the labour input, and $\boldsymbol{X}_{D(t)}$ is the vector $[x_{1,\ q(1)}, \ldots, x_{N(t),\ q(N(t))}]$ of the intermediate-goods quantities that participate in the production of y, according to $\boldsymbol{D}_t$. Our notation is justified by a set of assumptions implying that if $\boldsymbol{D}_t$ uses the variety v, then it uses only a well defined technology level $q(v)$ which is best-practice at t. Every activity ***D*** can be operated at a changing scale of operation a_D , under constant returns to scale and with the understanding that the unit scale of operation is such that $L_D = 1$. The output from the operation of ***D*** at scale a is $Y_D(a)$.

The input of the B activity for producing one unit of (v,q) is a quantity of capital K which depends positively on the technology level q_v. To fix our ideas, we may follow Aghion and Howitt ((1998), chapter 12) by assuming that K units of capital invested in the production of good (v, q) give rise to K/A^{qm} units of this good, thus implying that more capital intensive methods are required to produce intermediate goods of a later generation. m measures the sensitivity of production cost to a technological upgrade. For the sake of later reference we write:

$$K_{v,q} = x_{v,q} A^{qm} \qquad (1)$$

Howitt (1999) adopts a similar increasing-capital-intensity assumption and claims that capital used in intermediate-good production can be interpreted as human capital. The above specification implies that the average and marginal cost, in terms of final output, of producing (v, q) is rA^{qm} , where r is the rental price of capital. Since we abstract from depreciation, r is also the rate of interest.

An 'extensive' innovation is the introduction of a new variety v of initial technology level index any quality. An intensive innovation is the introduction of a subsequent generation q of an existing variety. Both kinds of innovation give rise to a new intermediate good (v,q), to the new activity producing it and to the new activities using (v,q), together with labour, and possibly other existing intermediate goods, as an input. Innovation is the non deterministic outcome of the research activity R performed by an economic agent i. The inputs of this activity are material and immaterial; the former represented by L_R, K_R, $\boldsymbol{X}_R$, the latter by the knowledge base of agent i, namely the set of goods Q_i of which i is knowledgeable.

2.1 Production activities

Even granting that the ordinal measure q is transformed in a cardinal productivity index, we should in general expect that the flow of service characteristics associated with (v, q) depends upon the type and quantity of other intermediate goods with which (v, q) co-operates within a production activity[1]. If there are strong complementarities between different intermediate goods, it may be the case that the best-practice technology level q of variety v at t may not be the last generation of v. Compatibility constraints may in fact

[1] If there are production externalities, this service flow may also depend upon the intermediate inputs participating in other production activities.

imply that it is inefficient to use in the same activity very distant technology levels of complementary varieties. Complementarities of his sort are simply ruled out in the R&D growth models.

As in standard microeconomic theory, the complementarity/substitutability relations between intermediate goods can be defined from the properties of the frontier of the production possibility set at time t, as synthesised by the production function $Y = F(L_{,t}\ ,\ \boldsymbol{X}_t)$. R&D growth models assume a particular substitutability relation between intermediate goods, to the effect that they enter the production function in the additively separable form:

$$Y_t = a\, N_t^{\gamma}\, L_{y,t}^{\,1-\alpha}\, [\sum_{v=1}^{N_t} \sum_{q=1}^{S_{v,t}} A^{q_v b}\ x_{v,q}^{\,\alpha}]$$

where a is a constant, $S_{v,t}$ is the latest generation of variety v at t. Thus, the marginal product of an intermediate good *(q,v)* is independent of the inputs of the other intermediate goods, although it may depend, if $\gamma \neq 0$, on the total *number* of intermediate goods potentially cooperating with *(q,v)*.

Since the last generation of a variety is equivalent to $A^{a/b}$ units of the previous generation, the price of the latter can not be higher than $1/A^{a/b}$ times the price of the former. Profit maximisation by the monopolist producing the last generation of v leads to a monopoly price[2] $p_t(S_{v,t}) = r_t\, A^{S_{v,t} m}\, a^{-1}$; thus the price at t of the last-but-one generation of v is $p_t(S_{v,t} - 1) \leq r_t\, A^{S_{v,t} m - b/a}\, a^{-1}$ against a marginal production cost $r_t\, A^{(S_{v,t}-1)m}$. Thus, the last-but-one or any older generation of v is not produced provided that $A^{m-b/a}\, a^{-1} < 1$. Since $\alpha < 1$, a sufficient condition is $\beta > \alpha m$ with A sufficiently large. Assuming that this holds true, the production function can be simplified to obtain:

$$Y_t = a\, N_t^{\gamma}\, L_{y,t}^{\,1-\alpha}\, [\sum_{v=1}^{N_t} A^{q_{v,t} b}\ x_v^{\,\alpha}] \qquad (2)$$

where $q_{v,\,t} = S_{v,\,t}$ and x_v is the quantity of the intermediate good (v, $q_{v,\,t}$).

The monopoly output $x_{v,\,t}$ of variety v is:

$$x_{v,\,t} = a^{1/(1-\alpha)} \alpha^{2/(1-\alpha)}\, N_t^{\gamma/(1-\alpha)}\, L_{y,t}\, r_t^{1/(\alpha-1)} [\, A^{q_{v,t}(b-m)/(1-a)}\,] \qquad (3)$$

Aghion and Howitt ((1998), chap. 12) obtain a monopoly output which is uniform across varieties and independent of q, by imposing the special assumption $\beta = m$. The more plausible restriction $b > m$ makes monopoly output positively related to the technological advance $q_{v,t}$.

Substitution of (3) into (2) yields:

$$Y_t = a^{\alpha/(1-\alpha)} \alpha^{2\alpha/(1-\alpha)}\, L_{y,t}\, r_t^{\alpha/(\alpha-1)}\, N_t^{(1-\alpha+\gamma)/(1-\alpha)}\, [(1/N_t) \sum_{v=1}^{N_t} A^{q_{v,t}(b-am)/(1-a)}\,]$$

(4)

[2] We may notice, in passing, that the pricing of last-generation varieties is uniform, provided that they share the same technology level; moreover, should we abstract from the influence of q on the intermediate-good-sector technology (m=0), the pricing of the last-generation varieties is uniform and independent of their technology level. This is the assumption we find in Barro and Sala I-Martin (1995), chapter 7.

Let $q_t = \sum_{v=1}^{N_t} q_{v,t}$. Realism would require that the distribution of $(q_{v,t} / q_t)$ be non uniform, and have a bounded or indefinitely increasing support, depending on the incentive to engage in intensive R&D in the relatively ‘backward’ or ‘advanced’ sectors and on the factors shaping the technology level of a new variety. We shall discuss below the different forces that are involved here. A convenient, if quite special case, arises when the probability distribution of a quality-innovation success is uniform across v and the technology level of a new variety is q_t. With this situation in mind, it is useful to write (4) in the following form:

$$Y_t = a^{\alpha/(1-\alpha)} \alpha^{2\alpha/(1-\alpha)} h_L L_{,t} r_t^{\alpha/(\alpha-1)} N_t^{(1-\alpha+\gamma)/(1-\alpha)} \cdot$$

$$M_t^{(b - am)/(1 - a)} [(1/N_t) \sum_{v=1}^{N_t} M_{v,t}^{(b - am)/(1 - a)} / M_t^{(b - am)/(1 - a)}] \tag{4.1}$$

Where h_L is the share of labour in the final-good sector, $M_t \equiv A^{q_t}$, $M_{v,t} \equiv A^{q_{v,t}}$ and the term in square brackets is easily shown to be constant through time and independent of N_t under the assumptions referred to above (see appendix A.1).

It is then clear how the assumption $\gamma = \alpha - 1$ (see, for instance, Aghion and Howitt (1998), chapter 12) sterilises the effects of the growing number of varieties on final output, which result from the additively separable way in which the single varieties enter the production function. Where these effects are not sterilised, because $(1 - \alpha + \gamma) > 0$, we observe that the production function corresponding to a constant technology level contains a form of increasing returns due to specialisation, as measured by N. The best known example along these lines is probably Romer (1990), which assumes $\gamma = 0$.

In every sector v the expected proportional change per unit of time of the term $M_{v,t}$ is the proportional change in this term due to an innovation success times the probability P_t of an innovation arrival in period t[3]. Since the proportionate change of $M_{v,t}$ due to an innovation success is $[A - 1]$ and P_t is assumed to be uniform across v we obtain:

$$E(\Delta M_{v,t} / M_{v,t}) = E(\Delta M_t / M_t) = P_t [A - 1] \tag{5}$$

If the number of varieties N_t is sufficiently large M_t has a deterministic time rate of change $g_M(t)$ equal to the right-hand side of (4).

Recalling that in steady state the rate of interest is constant, equation (4.1) yields the steady-state-growth approximation:

$$g_Y = g_L + [(1 - a + g)/(1 - a)]\, g_N. + [(b - am)/(1 - a)] g_M \tag{6}$$

where g_i is the proportional rate of change of variable i in a unit time interval[4]. In particular, if following Romer (1990) we impose the restrictions $\gamma = 0$ and $g_M = 0$, the above relation boils down to $g_y = g_L + g_N$, where it is apparent that the growth rate of per-capita output is simply the growth rate in the number of specialised varieties.

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[3] Cf. Barro and Sala I-Martin (1995), p. 252.

[4] The relation holds strictly in continuous time.

The same steady-state-growth equation is obtained by considering the fictitious economy where the technology level $q_{v,t}$ of each variety v equals the average technology level of the real economy q_t. The main difference between the real and the fictitious economy is that in the former, but not in the latter, the proportional time rate of change of quality advance in every single sector is stochastic, with an expected value equal to the deterministic value we observe in the fictitious economy.

In the fictitious economy monopoly output is uniform across varieties ($x_{v,\,t} = x_t$) and final output can be expressed:

$$Y_t = a\, N_t^{\gamma}\, L_{y,t}^{\,1-\alpha}\, N_t\, A^{q_t b}\, x_t^{\alpha}$$

If h_K is the capital share employed in the production, as opposed to the innovation, activities, it must be the case that, in equilibrium $h_{K,\,t}\, K_t \,/\, A^{q_t m} = N_t\, x_t$. On the symmetric-equilibrium path of the fictitious economy final output is then:

$$Y_t = a\, N_t^{\gamma}\, (h_{L,\,t}\, L_t)^{\,1-\alpha}\, N_t^{1-\alpha}\, M_t^{(b\,-\,am)}\, (h_{K,\,t}\, K_t)^{\alpha} \qquad (7)$$

This yields the steady-state approximation (6), because the shares h_L and h_K are constant on a steady-state path.

2.3 Intensive innovations

The argument above rests on the assumption that the probability of an innovation arrival is uniform across varieties. It is now time to discuss the plausibility of the assumption. The probability $P_{v,\,q}$ that in a given period an intensive-innovation success occurs in sector v with technology level q depends on the rival and non rival resources invested in R&D in sector v and on the complexity of the search effort. The rival resources are represented by K_v and L_v , the non-rival ones by the sector specific knowledge stock and the general knowledge stock made available by technological advances in other sectors.

The drastic simplification we find in the literature is that these knowledge stocks can be conceived as scalar multiples of homogeneous magnitudes; in particular, they can be measured by the sector technology level q_v and the average technology level q, respectively. Thus, the same hyper-simplified framework used to measure the contribution of knowledge to final output production is transferred to evaluate the contribution of knowledge to the production of knowledge. We shall not dwell here on the potential dangers of this hyper-simplification. More concretely, we shall insist, here and in later sections, on the directions in which the framework should be expanded, in order that some basic phenomena, such as the relation between innovation and the difficulty of search, or knowledge obsolescence, or the complementarity between sector specific knowledge stocks, can be brought within the scope of the theory. For the time being, we resort to the following specification of the determinants of $P_{v,\,q}$.

$$P_{v,\,q} = l_Q\; f(u_{L,v}\, L,\, u_{K,v}\, K,\, M_v,\, M) = l_Q\, (u_{L,\,v}\, L)^{\theta}\, (u_{K,v}\, K)^{\xi} M_v^{\chi}\, [M^{\varphi} / M_v^{\varphi\omega}] \qquad (8)$$

where $\xi > 0$, $\theta > 0$, l_Q is a constant, $u_{L,\,v}$, $u_{K,\,v}$ are the fractions of total labour and capital invested in intensive R&D on variety v. The returns of the R&D activity with respect to rival-resource investment are constant or decreasing, depending on $\theta + \xi = 1$ or $\theta + \xi < 1$. The second case arises if there is a congestion effect on the returns to R&D investment (Stokey (1995), Howitt (1999)).

It may be worth observing that $u_v L$ can be interpreted as a quantity of human capital in the sense that u_v includes the fraction of time spent for attaining and maintaining qualification. This interpretation of the above formalism requires that the depreciation rate of human capital is close to 1, perhaps as a result of knowledge obsolescence.

The parameter χ is meant to capture how the arrival probability is affected by sector-specific knowledge for reasons independent of cross-sector (horizontal) knowledge spill-overs. The sign of χ is highly controversial, because there are at least three different ways in which q_v may affect the success probability $P_{v,\ q}$. Equation (3) suggests how a relatively large q_v, hence M_v, multiplies the benefit of an innovation success over a relatively large scale of monopoly output (Barro and Sala I-Martin (1995), chapter 7). This channel of influence makes research investment in the advanced sectors more profitable, the equilibrium value of u_v positively correlated with M_v and the probability that a technology advance occurs in a relatively advanced sector higher.

A powerful influence acting in the opposite direction arises if more advanced technology levels are progressively more difficult to discover as a result of the increasing complexity of the search activity. This is the assumption we find in a number of search-theoretic models of R&D-based economic growth (Jovanovic and Rob (1990), Stokey (1995), Kortum (1997))[5]. Realistic as it may be, the assumption does not find a theoretical justification *within* a formal framework which postulates a fixed search space with a fixed (and indeed very simple) local structure. The feed-back of innovations on the complexity of the local search space is therefore beyond the reach of these model and must be imported from the outside, as an exogenous fact. There is eventually the "standing on giants' shoulders' effect"[6] (Caballero and Jaffe (1993)) which may act *within* a sector. The hypothesis is here that, abstracting from cross-sector spill-overs, a higher *sector-specific* knowledge facilitates discovery. It should be observed how an effect of this kind is already captured by the very notion of a 'quality ladder' (Grossman and Helpman (1991)). In the present framework, this is reflected in the assumption that the innovation with arrival probability $P_{v,\ q}$ shifts the best-practice technology level from q_v to $q_v + 1$. Thus, the probability that $q_v + 1$ is discovered is zero if the starting technology level is lower than q_v. Moreover, since the productivity parameter is $M_{v,\ t} \equiv A^{q_{v,t}}$, the idea of a quality-ladder amounts to an intertemporal spill-over, to the effect that a quality innovation now increases the productivity gain associated with future innovations. The question is therefore about the grounds for including, *on this account*, a further positive influence of q_v on $P_{v,q}$.

The above issue must be distinguished from the effect that sector-specific knowledge may have on the sign and strength of *cross-sector* knowledge externalities, or aggregate "giants' shoulders effect". These effects are captured by the parameters φ and ω. The hypothesis is that intensive R&D in a sector is exposed

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[5] In view of their rather special structure, Aghion and Howitt (1998) and Howitt (1999) refer this complexity-of-search effect to the maximum technology level in the economy, rather than to the sector-specific q_v.

[6] Newton …

to positive or negative externalities, depending on the relatively backward or advanced technological position of the sector. If the externalities are positive ($M > M_v$), they can be simply interpreted as cross-sector knowledge spill-overs; if they are negative ($M < M_v$), they may arise from the inertia that a backward technological environment exerts on the pace of progress, or from problems of technological incompatibility. This leads to $\varphi > 0$. The parameter $\omega > 0$ measures the relative strength of positive and negative externalities, with $\omega = 1$ referring to a situation of equal strength, so that the effect vanishes for a sector with average technology level q (for an equivalent assumption, see Peretto (1998)). It may be also worth observing that the above restrictions disregard that being *too* backward may sometimes inhibit from the possibility of taking advantage of knowledge spillovers[7].

Expression (8) is equivalently written as follows:

$$P_{v,q} = l_Q\,(u_{L,v}\,L)^{\theta}\,(u_{K,v}\,K)^{\xi}\,M_v^{\chi-\omega\varphi}\,M^{\varphi} \qquad (8')$$

It is of course very difficult to assess the relative weight of the different channels of influence of q_v on $P_{v,q}$ and therefore the sign of χ and of $\chi - \omega\varphi$. In models where the different effects of the sector specific q_v on R&D productivity exactly cancel out (for instance, Barro and Sala I-Martin (1995), chapter 7 assumes $\chi = 0$, $\varphi = 0$) the arrival probability $P_{v,q}$ is independent of q_v. This implies that the arrival probability is uniform across sectors in the presence of a uniform distribution of labour invested in research. Referring for simplicity to growth paths with a constant interest rate, we obtain that the gross expected return on R&D investment is $r + P_{v,q}$, where $P_{v,q}$ can be also interpreted as the premium for the probability that the flow of monopoly profit associated with the innovation stops in finite time (see appendix A.2). Thus the rate of return on R&D investment is uniform across sectors when the distribution of R&D labour, and therefore the arrival probability $P_{v,q}$ is also uniform. Still, since innovation arrival in every single sector is stochastic, the support of the relative technology-level distribution $q_{v\ t} / q_t$ grows with time, because the sectors where the realization of technology arrival has been relatively slow face the same arrival probability $P_{v,q}$ faced by the relatively advanced sectors.

If instead $\chi - \omega\varphi < 0$, the relatively backward sectors have a tendency to catch up (in terms of technology level) with the relatively advanced ones (see appendix A.2). In each sector, the ratio $q_{v\,t} / q_t$ converges to 1 and the real economy converges to the fictitious economy where $q_v = q$ and intensive R&D investment is uniform across varieties every date t. As will be shown, the situation is consistent with a sustained growth of the average technology level q_t provided that $\omega < 1$ (positive cross-sector externalities stronger than negative cross sector externalities).

2.4 Extensive innovations

On the assumption that there is an external effect such that the technical knowledge in the economy affects the technology level of a new variety, a-not-too-unplausible assumption is that if (v, q_v) is a new variety born at t, then q_v is stochastic, with distribution determined by the cross-sector distribution of q at t

[7] To include this possibility, it should be allowed that the sign of ω may depend on M / M_v.

(Howitt (1999)). For the sake of simplicity, we shall assume instead that the technology level of a new variety born at t is q_t (Peretto (1998)). The hypothesis brings in a certain symmetry between extensive and intensive innovations, in that for both of them the technology level of the new intermediate good produced by the innovation is deterministic. Moreover, extensive innovations, as it happens for the intensive ones, face some risk of being displaced by a future quality improvement. In steady-state equilibrium, the gross rate of return on the extensive innovation *(v, q)* is $r + P_{v,\, q}$, where the second term is the risk premium demanded to compensate for the probability that a future innovation *(v, q +* 1*)* displaces the intermediate good *(v, q)*. In fact, if $P_{v,\, q}$ does not depend on q (because $\chi = 0$) and is uniform across v, then all innovations, no matter whether intensive or extensive, face the same risk of being displaced in the future.

Our simplifying hypothesis on the technology level of a new variety does not eliminate the uncertainty about the success or failure of the *individual* extensive R&D investment in a given time period. In the economy as a whole, the number of extensive innovations in a given period is still random, and their expected number is here expressed as:

$$DN = l_{\mathrm{N}}\, (z_L\, L)^{e} (z_K\, K)^{\psi} N^{\tau} M^{n} \qquad (9)$$

where l_{N} is a constant, z_L and z_K are the fractions of total labour and capital employed in extensive R&D; again, z_L includes the fraction of labour time invested in qualification. We impose the restrictions $\varepsilon > 0$ and $\psi > 0$, with $\varepsilon + \psi < 1$ indicating that there is a congestion effect on the productivity of research. A positive τ bears the interpretation that a higher number of varieties amounts to a wider knowledge base in the economy as a whole and therefore facilitates the discovery of yet new varieties. If this is in itself quite plausible, far more questionable appear to be 'point restrictions' such as $\tau = 1$, or $\tau = 0$, as may be found, for instance, in the pure variety-extension model of Romer (1990) and in Peretto (1998), respectively. It may be also worth observing that the condition $\nu = 0$ can be interpreted in the sense that the spill-over from the average quality q (hence, the average productivity parameter M) is just sufficient to make sure that the new varieties are of average quality, and there is no further spill-over effect from q (Peretto (1998)). $\nu < 0$ would instead indicate that the cost (in terms of rival resources invested in extensive R&D) of producing a given innovation flow DN with average technology level q is increasing in q.

2.5 Steady-growth equations

For the sake of simplicity we refer below to the symmetric equilibrium where $q_v = q$ and $M_v = M$. We have shown that if $\chi - \omega\varphi < 0$ the economy has a tendency to converge to such equilibria. Thus, our simplification amounts to the hypothesis that convergence has already taken place. It has been also shown that if $\chi - \omega\varphi = 0$ the output-growth properties of the fictitious economy with $M_v = M$ are descriptive of the growth properties of the real economy under the restriction that the rate of interest is constant.

For the sake of later reference, we introduce the notion of a *constant-growth path* as defined by following properties: *(i)* all the variables, including the factor employment shares, grow at a constant rate, possibly equal to zero; *(ii)* the capital output ratio K/Y is constant. A steady state, or balanced-growth path, is a particular constant-growth path such that the growth rate of every variable is constant *for ever*. Since the factors employment shares can not exit the interval [0, 1], the definition immediately implies that the growth rate of such variables is zero on a balanced path. The reason for introducing the notion of a constant-growth path is related to a number of stylised facts on economic growth in the advanced countries over the last one hundred years. In particular, the dramatic rise in the labour's share employed in research is prima facie inconsistent with the properties of a steady-state path, but may fit those of a constant-growth path (Jones (2000)).

On a symmetric-equilibrium path equations (5) and (8') yield:

$$\Delta M = [A - 1]\ l_Q\, (u_L\, L/N)^{\theta}\, (u_K\, K/N)^{\xi}\, M^{\,1 + \chi + \varphi(1-\omega)} \qquad (10)$$

where u_L and u_K are the aggregate labour and capital shares invested in intensive R&D in the N sectors.

As discussed in Appendix A.3, most models of endogenous growth with intensive R&D introduce ad-hoc assumptions that, when translated in terms of our notation, amount to the very special case: $\chi + \varphi(1 - \omega) = 0$ and $\xi = 0$ (see, for instance, Grossman and Helpman (1991), Aghion and Howitt (1992), Howitt (1999), Peretto (1998), Young (1998), Barro and Sala I-Martin (1995), chapter 7). This implies that $\Delta M / M = [A - 1]\ l_Q\, (u_L\, L/N)^{\theta}$. In particular, in the models where N is constant, it is assumed that L is also constant. If instead, the number of varieties grows simultaneously with the technology level, special assumptions (to be considered below) make sure that L/N is constant *at least in steady state* (Howitt (1999), Peretto (1998), Young (1998)). In either case, the steady-state growth rate of M is positive provided that a positive labour share u_L is *permanently* invested in intensive R&D[8]. Indeed, the balanced-growth rate g^*_M is then an increasing function of the steady-state proportion u^*_L *which depends on preferences.* We shall see below how $\chi + \varphi(1 - \omega) = 0$ is necessary to the last result, which, in the light of our discussion in section 2.3, can be regarded as the outcome of a prohibitively special restriction.

On a constant-growth path ΔM and M grow at the same rate; thus, on the assumption that $\chi+\varphi(1-\omega) \neq 0$, equation (10) implies that on a constant-growth path:

$$g_M [-\chi - \varphi(1-\omega)] = [\theta\,(n + g_{u_L}) + \xi\,(g_{u_K} + g_K) - (\xi + \theta)\, g_N] \qquad (11)$$

The corresponding expressions for $g_Y = g_K$ and g_N are as follows:

$$g_K\,(1-\alpha) = (\gamma + 1 - \alpha)\, g_N + (1-\alpha)\,(\mathrm{n} + g_{h_L}) + \alpha\, g_{h_K} + (\beta - \alpha\mu)\, g_M \qquad (12)$$

$$g_N\,(1-\tau) = \varepsilon\,(n + g_{z_L}) + \psi\,(g_{z_K} + g_K) + \nu\, g_M \qquad (13)$$

In a resource-full-employment equilibrium the growth rates of the factor shares are constrained by the equlibrium condition $h_i + z_i + u_i = 1$ for $i = K, L$. In particular, on a balanced-growth path the growth rates of the factor shares are all zero, so that system (11), (12), (13) can be specialised as follows:

$$g_M [-\chi - \varphi(1-\omega)] + (\xi + \theta)\, g_N - \xi\, g_K = \theta\, n \qquad (14)$$

[8] This may well be the case even if there is a congestion effect in research so that $0 < \theta < 1$ (Howitt (1999)).

$$-\nu\, g_M + (1-\tau)\, g_N - \psi\, g_K = \varepsilon\, n \tag{15}$$

$$-(\beta-\alpha\mu)\, g_M - (\gamma+1-\alpha)\, g_N + g_K\,(1-\alpha) = (1-\alpha)\, n \tag{16}$$

If we define the variables $k \equiv K/N$, $l \equiv L/N$, so that $g_K = g_k + g_N$, $n = g_l + g_N$, (14) – (15) – (16) yield the following system:

$$\begin{bmatrix} -\chi-\varphi(1-\omega) & 0 & -\xi \\ -\nu & 1-\tau-\varepsilon-\psi & -\psi \\ -(\beta-\alpha\mu) & -(\gamma+1-\alpha) & 1-\alpha \end{bmatrix} \begin{bmatrix} g_M \\ g_N \\ g_k \end{bmatrix} = \begin{bmatrix} \theta g_l \\ \varepsilon g_l \\ (1-\alpha) g_l \end{bmatrix} \tag{17}$$

Let $[I - \Gamma]$ be the square matrix in the left-hand-side of (17). We obtain the following proposition which extends to the economy with expanding varieties and technology levels a result, similar in spirit, in Eicher and Turnovsky (1999).

Proposition 2.1: Assume $\Gamma \geq 0$ and Trace $([I - \Gamma]) > 0$. Assume also that, for each row, the row sum of the elements of Γ is positive and lower than 1. Then, for every $n > 0$, there exist positive values g_M, g_N, g_K that are solutions to (14)-(15)-(16) and such that $g_l = n - g_N > 0$.

Proof: see appendix A.2.

A quick look at equation (16) will suffice to see that the following holds:

Proposition 2.2: If, in addition to the assumptions of proposition 2.1, either or both elements $(\beta - \alpha\mu)$ and $(\gamma + 1 - \alpha)$ in the third row of Γ are positive, then $g_K > n$ (positive per-capita-output growth).

Remark 2.1: The *if* condition of proposition 2.2 amounts to the existence of increasing returns to scale in the output sector. Proposition 2.1 requires instead that there are decreasing returns to scale in extensive search and is consistent with decreasing, constant or increasing returns in intensive search. In particular , the growth of the productivity index *M* will be faster than population growth, provided that θ is sufficiently large.

We have reached the remarkable conclusion that under the conditions of proposition 2.1 the steady-state growth rates of output, technology levels and varieties are completely determined by the exogenous population growth and technological parameters. They are therefore independent of preferences, and of savings rates in particular. The reason why the same conclusion fails to hold in many R&D models of endogenous growth is that they assume ad-hoc conditions which make sure that the coefficients of equations (14) and (15) are linearly dependent, hence that the determinant of $[I - \Gamma]$ is zero (see Appendix A.3). It may be worth observing how this amounts to a choice of parameters which has measure zero in the relevant parameter space. If the conditions imposed by proposition 2.1 are certainly less restrictive, it may not follow that they are economically realistic. In particular, it is required that $\chi + \varphi(1 - \omega) < 0$. Thus, recalling our discussion in section 2.2 and the economic interpretation of the parameters χ, φ and ω, we may observe how this condition amounts to the fact that the real economy does indeed converge to the situation in which the technology level is uniform across sectors, because the relatively backward sectors improve faster than the relatively advanced ones. This may be either because $\chi < 0$ and ω is sufficiently close to 1, or because $\chi > 0$ and ω is sufficiently larger than 1.